Refutation of Lockyer's Argument Against my Disproof of Bell's Theorem

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I identify the mistake in Lockyer's repeated claim of a sign error in my disproof of Bell's theorem. In particular, I point out that his argument is grounded, not only on a misreading of my central equation, but also on an oversight of a freedom of choice in the orientation of a geometric algebra.

Lockyer has repeatedly claimed online that there is a sign error in my equation (17) below. To bring out his mistake, consider a right-handed frame of ordered basis bivectors, $\{\boldsymbol{\alpha}_x,\,\boldsymbol{\alpha}_y,\,\boldsymbol{\alpha}_z\}$, and the corresponding bivector sub-algebra

$$\boldsymbol{\alpha}_i\,\boldsymbol{\alpha}_j \,=\, -\,\delta_{ij}\,-\,\epsilon_{ijk}\,\boldsymbol{\alpha}_k \tag{1}$$

of the Clifford algebra $Cl_{3,0}$. The latter is a vector space, $\mathrm{I\!R}^8$, spanned by the ordered set of graded orthonormal basis

$$\{1,\;\mathbf{e}_x,\,\mathbf{e}_y,\,\mathbf{e}_z,\;\mathbf{e}_y\wedge\mathbf{e}_z,\,\mathbf{e}_z\wedge\mathbf{e}_x,\,\mathbf{e}_x\wedge\mathbf{e}_y,\;\mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z\}, \tag{2}$$

where δ_{ij} is the Kronecker delta, ϵ_{ijk} is the Levi-Civita symbol, the indices $i,\,j,\,k=x,\,y,$ or z are cyclic indices, and

$$\boldsymbol{\alpha}_i \,=\, \mathbf{e}_j\wedge\mathbf{e}_k \,=\, I\cdot\mathbf{e}_i\,, \tag{3}$$

with $I:=\mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z$ being a volume element of physical space. Eq. (1) is a standard definition of bivector subalgebra, routinely used in geometric algebra [1]. From it, it is easy to verify the basic properties of the basis bivectors, such as

$$(\boldsymbol{\alpha}_x)^2 \,=\, (\boldsymbol{\alpha}_y)^2 \,=\, (\boldsymbol{\alpha}_z)^2 \,=\, -\,1 \tag{4}$$

$$\text{and}\quad \boldsymbol{\alpha}_x\,\boldsymbol{\alpha}_y \,=\, -\,\boldsymbol{\alpha}_y\,\boldsymbol{\alpha}_x \quad \text{etc.} \tag{5}$$

Moreover, it is easy to verify that the bivectors satisfying the subalgebra (1) form a right-handed frame of basis bivectors. To check this, right-multiply both sides of Eq. (1) by $\boldsymbol{\alpha}_k$, and then use the fact that $(\boldsymbol{\alpha}_k)^2=-1$ to arrive at

$$\boldsymbol{\alpha}_i\,\boldsymbol{\alpha}_j\,\boldsymbol{\alpha}_k \,=\, +\,1\,. \tag{6}$$

The fact that this ordered product yields a positive value confirms that $\{\boldsymbol{\alpha}_x,\,\boldsymbol{\alpha}_y,\,\boldsymbol{\alpha}_z\}$ indeed forms a right-handed frame of basis bivectors. This is a universally accepted convention, easily found in any textbook on geometric algebra.

Suppose now $\mathbf{a}=a_i\,\mathbf{e}_i$ and $\mathbf{b}=b_j\,\mathbf{e}_j$ are two unit vectors in $\mathrm{I\!R}^3$, where the repeated indices are summed over x, y, and z. Then the right-handed set of graded basis defined in Eq. (1) leads to

$$\{\,a_i\,\boldsymbol{\alpha}_i\,\}\,\{\,b_j\,\boldsymbol{\alpha}_j\,\} \,=\, -\,a_i\,b_j\,\delta_{ij}\,-\,\epsilon_{ijk}\,a_i\,b_j\,\boldsymbol{\alpha}_k\,, \tag{7}$$

which, together with (3) (which says that both $+\mathbf{e}_i$ and $\boldsymbol{\alpha}_i$ form right-handed frames), is equivalent to the identity

$$(\,I\cdot\mathbf{a})(\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b}\,-\,I\cdot(\mathbf{a}\times\mathbf{b}), \tag{8}$$

where $I=\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z$ is the standard trivector. Geometrically this identity describes all points of a parallelized 3-sphere.

Let us now consider a left-handed frame of ordered basis bivectors, which we denote by $\{\boldsymbol{\beta}_x,\,\boldsymbol{\beta}_y,\,\boldsymbol{\beta}_z\}$. It is important to recognize, however, that there is no *a priori* way of knowing that this new basis frame is in fact left-handed. To ensure that it is indeed left-handed we must first make sure that it is an ordered frame by requiring that its basis elements satisfy the bivector properties analogous to those delineated in Eqs. (4) and (5). Next, to distinguish this frame from the right-handed frame defined by equation (6), we must require that its basis elements satisfy the property

$$\boldsymbol{\beta}_i\,\boldsymbol{\beta}_j\,\boldsymbol{\beta}_k \,=\, -\,1\,. \tag{9}$$

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One way to ensure this is by multiplying all vector and bivector elements in the basis set (2) by a minus sign, giving

$$\{1,\; -\mathbf{e}_x,\, -\mathbf{e}_y,\, -\mathbf{e}_z,\; -\mathbf{e}_y\wedge\mathbf{e}_z,\, -\mathbf{e}_z\wedge\mathbf{e}_x,\, -\mathbf{e}_x\wedge\mathbf{e}_y,\; \mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z\}. \tag{10}$$

Note that we have not changed the signs of the scalar 1 and the pseudoscalar $I := \mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z$. As a result, although the four-dimensional even and odd subalgebras are now left-handed, the full eight-dimensional algebra remains right-handed, because we have changed the signs of only an *even* number of its elements, namely three vectors plus three bivectors, comprising six elements in total. Another way to see this is by noting that the determinant of the matrix that transforms the basis (2) into (10) is $(-1)^6 = +1$. On the other hand, instead of the relationship (3) we now have

$$\boldsymbol{\beta}_i \,=\, -\,\mathbf{e}_j\wedge\mathbf{e}_k \,=\, I\cdot(\,-\,\mathbf{e}_i\,)\,, \tag{11}$$

and therefore the condition (9) above is automatically satisfied. As is well known, this was the condition imposed by Hamilton on his unit quaternions, which we now know are nothing but a left-handed set of basis bivectors. Indeed, it can be easily checked that the basis bivectors satisfying the properties (4), (5), (9), and (11) compose the subalgebra

$$\boldsymbol{\beta}_i\,\boldsymbol{\beta}_j \,=\, -\,\delta_{ij} \,+\, \epsilon_{ijk}\,\boldsymbol{\beta}_k\,. \tag{12}$$

Suppose now $\mathbf{a} = a_i\,\mathbf{e}_i$ and $\mathbf{b} = b_j\,\mathbf{e}_j$ are two unit vectors in $\mathrm{I\!R}^3$, identical to those used in Eq. (7), where the repeated indices are again summed over x, y, and z. Then the left-handed set of graded basis defined in (12) leads to

$$\{\,a_i\,\boldsymbol{\beta}_i\,\}\,\{\,b_j\,\boldsymbol{\beta}_j\,\} \,=\, -\,a_i\,b_j\,\delta_{ij} \,+\, \epsilon_{ijk}\,a_i\,b_j\,\boldsymbol{\beta}_k\,, \tag{13}$$

which, together with (11) (which says that both $-\mathbf{e}_i$ and $\boldsymbol{\beta}_i$ form left-handed frames), is equivalent to the identity

$$(\,I\cdot\mathbf{a})(\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, I\cdot(\mathbf{a}\times\mathbf{b}), \tag{14}$$

where I is the standard trivector. Once again, geometrically this identity describes all points of a parallelized 3-sphere.

Note that the geometric identities (8) and (14) are identical despite the fact that the bivector relations (7) and (13) are not. Thus, unlike the cross product, the geometric product between bivectors remains invariant under orientation changes if they are confined to the even (*i.e.*, bivector) and odd (*i.e.*, vector) subalgebras. But suppose we consider now orientation change in the entire algebra $Cl_{3,0}$ of the orthogonal directions in the 3D space by means of the basis

$$\{1,\; -\mathbf{e}_x,\, -\mathbf{e}_y,\, -\mathbf{e}_z,\; -\mathbf{e}_y\wedge\mathbf{e}_z,\, -\mathbf{e}_z\wedge\mathbf{e}_x,\, -\mathbf{e}_x\wedge\mathbf{e}_y,\; -\mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z\}, \tag{15}$$

where the sign of every non-scalar element is now different from that in the set (2), including the volume element I:

$$I \longrightarrow -I := -\,\mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z\,. \tag{16}$$

Since the determinant of the matrix that transforms (2) into (15) is $(-1)^7 = -1$, the basis defined by (15) is genuinely left-handed *relative* to the basis defined by (2). The question then is: How do the identities (8) and (14) transform into one another under the handedness transformation (16) of the volume element? It is not difficult to see that under (16) the identity (14) transforms into

$$(\,I\cdot\mathbf{a})(\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,+\, I\cdot(\mathbf{a}\times\mathbf{b}). \tag{17}$$

Crucially, there is a sign difference in the second term on the RHS of the above identity compared to the identity (8). Consequently, the identities (8) and (17) now transform into one another under the handedness transformation (16):

$$(\,I\cdot\mathbf{a})(\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, I\cdot(\mathbf{a}\times\mathbf{b}) \;\xleftrightarrow{+I\,\longleftrightarrow\,-I}\; (\,I\cdot\mathbf{a})(\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,+\, I\cdot(\mathbf{a}\times\mathbf{b}). \tag{18}$$

For convenience, we can now rewrite these two alternative identities (8) and (17) as two hidden variable possibilities

$$(\,+I\cdot\mathbf{a})(\,+I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, (\,+I\,)\cdot(\mathbf{a}\times\mathbf{b}) \tag{19}$$

and

$$(\,-\,I\cdot\mathbf{a})(\,-\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, (\,-\,I\,)\cdot(\mathbf{a}\times\mathbf{b}). \tag{20}$$

Exploiting the natural freedom of choice in characterizing orientation of the 3-sphere by either $+\,I$ or $-\,I$, we can now combine the identities (19) and (20) into a single hidden variable equation (at least for the computational purposes):

$$(\,\lambda\,I\cdot\mathbf{a})(\,\lambda\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, (\,\lambda\,I\,)\cdot(\mathbf{a}\times\mathbf{b}), \tag{21}$$

where $\lambda = \pm 1$ now specifies the orientation of the 3-sphere. It is important to keep in mind here that the combined equation (21) is simply a convenient shortcut for representing two completely independent initial states of the physical system, one corresponding to the counterclockwise orientation of the 3-sphere and the other corresponding to the clockwise orientation of the 3-sphere. Moreover, at no time are these two alternative possibilities mixed during the course of an experiment. They represent two independent physical scenarios, corresponding to two independent runs of the experiment. Next, if we employ the notation $\boldsymbol{\mu} = \lambda I$, then the combined identity (21) takes the convenient form

$$\boxed{(\,\boldsymbol{\mu}\cdot\mathbf{a})(\,\boldsymbol{\mu}\cdot\mathbf{b}) \;=\; -\,\mathbf{a}\cdot\mathbf{b} \;-\; \boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b}).} \tag{22}$$

This is the central equation of my local model. I have used it in various forms and notations since 2007 [2]. It is simply an isomorphic representation of the familiar identity from quantum mechanics, with the correspondence $\boldsymbol{\mu} \longleftrightarrow i\boldsymbol{\sigma}$:

$$(i\boldsymbol{\sigma}\cdot\mathbf{a})(i\boldsymbol{\sigma}\cdot\mathbf{b}) \;=\; -\,\mathbf{a}\cdot\mathbf{b}\,\mathbb{1} \;-\; i\,\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b}). \tag{23}$$

In Ref. [3] I have combined the two alternative bivector relations (7) and (13) into a single hidden variable equation

$$L_\mu(\lambda)\,L_\nu(\lambda) \;=\; -\,\delta_{\mu\nu} \;-\; \sum_\rho \epsilon_{\mu\nu\rho}\,L_\rho(\lambda)\,, \tag{24}$$

together with $L_\mu(\lambda) := \lambda\,D_\mu$, with alternative choices $\lambda = \pm 1$ for the orientation of S^3. Contracting this equation on both sides with the components a^μ and b^ν of arbitrary unit vectors $\mathbf{a}$ and $\mathbf{b}$ then gives the convenient bivector identity

$$\mathbf{L}(\mathbf{a},\,\lambda)\,\mathbf{L}(\mathbf{b},\,\lambda) \;=\; -\,\mathbf{a}\cdot\mathbf{b} \;-\; \mathbf{L}(\mathbf{a}\times\mathbf{b},\,\lambda)\,, \tag{25}$$

which is a convenient notation for the identity (22). It combines the alternative identities (8) and (17) into a single identity, rendering the unit bivector $\mathbf{L}(\mathbf{n},\,\lambda^k)$ a random variable *relative* to the detector bivector $\mathbf{D}(\mathbf{n})$, for a given run:

$$\mathbf{L}(\mathbf{n},\,\lambda^k) \;=\; \lambda^k\,\mathbf{D}(\mathbf{n}) \;\Longleftrightarrow\; \mathbf{D}(\mathbf{n}) \;=\; \lambda^k\,\mathbf{L}(\mathbf{n},\,\lambda^k)\,. \tag{26}$$

The expectation value of simultaneous outcomes $\mathscr{A}(\mathbf{a},\,\lambda^k) = \pm 1$ and $\mathscr{B}(\mathbf{b},\,\lambda^k) = \pm 1$ in S^3 then works out as follows:

$$\begin{aligned}\mathcal{E}(\mathbf{a},\,\mathbf{b}) \;&=\; \lim_{n\,\rightarrow\,\infty}\left[\frac{1}{n}\sum_{k\,=\,1}^{n}\mathscr{A}(\mathbf{a},\,\lambda^k)\;\mathscr{B}(\mathbf{b},\,\lambda^k)\right] \;=\; \lim_{n\,\rightarrow\,\infty}\left[\frac{1}{n}\sum_{k\,=\,1}^{n}\mathbf{L}(\mathbf{a},\,\lambda^k)\,\mathbf{L}(\mathbf{b},\,\lambda^k)\right] \\ &=\; \frac{1}{2}\Big[\mathbf{L}(\mathbf{a},\,\lambda^k=+1)\;\mathbf{L}(\mathbf{b},\,\lambda^k=+1)\Big] \;+\; \frac{1}{2}\Big[\mathbf{L}(\mathbf{a},\,\lambda^k=-1)\;\mathbf{L}(\mathbf{b},\,\lambda^k=-1)\Big],\end{aligned} \tag{27}$$

where the last simplification occurs because λ^k is a fair coin. Using the relations (25) and (26) the above sum can now be evaluated by recognizing that the spins in the right and left oriented S^3 satisfy the following geometrical relations:

$$\begin{aligned}\mathbf{L}(\mathbf{a},\,\lambda^k=+1)\;\mathbf{L}(\mathbf{b},\,\lambda^k=+1) \;&=\; -\,\mathbf{a}\cdot\mathbf{b} \;-\; \mathbf{L}(\mathbf{a}\times\mathbf{b},\,\lambda^k=+1) \\ &=\; -\,\mathbf{a}\cdot\mathbf{b} \;-\; \mathbf{D}(\mathbf{a}\times\mathbf{b}) \\ &=\; \mathbf{D}(\mathbf{a})\;\mathbf{D}(\mathbf{b}) \;=\; (\,+I\cdot\mathbf{a})(\,+I\cdot\mathbf{b})\end{aligned} \tag{28}$$

and

$$\begin{aligned}\mathbf{L}(\mathbf{a},\,\lambda^k=-1)\;\mathbf{L}(\mathbf{b},\,\lambda^k=-1) \;&=\; -\,\mathbf{a}\cdot\mathbf{b} \;-\; \mathbf{L}(\mathbf{a}\times\mathbf{b},\,\lambda^k=-1) \\ &=\; -\,\mathbf{a}\cdot\mathbf{b} \;+\; \mathbf{D}(\mathbf{a}\times\mathbf{b}) \\ &=\; -\,\mathbf{b}\cdot\mathbf{a} \;-\; \mathbf{D}(\mathbf{b}\times\mathbf{a}) \\ &=\; \mathbf{D}(\mathbf{b})\;\mathbf{D}(\mathbf{a}) \;=\; (\,+I\cdot\mathbf{b})(\,+I\cdot\mathbf{a}).\end{aligned} \tag{29}$$

In other words, when λ^k happens to be equal to $+1$, $\mathbf{L}(\mathbf{a},\,\lambda^k)\;\mathbf{L}(\mathbf{b},\,\lambda^k) = (\,+I\cdot\mathbf{a})(\,+I\cdot\mathbf{b})$, and when λ^k happens to be equal to -1, $\mathbf{L}(\mathbf{a},\,\lambda^k)\;\mathbf{L}(\mathbf{b},\,\lambda^k) = (\,+I\cdot\mathbf{b})(\,+I\cdot\mathbf{a})$. Consequently, the expectation value (27) reduces at once to

$$\mathcal{E}(\mathbf{a},\,\mathbf{b}) \;=\; \frac{1}{2}(\,+I\cdot\mathbf{a})(\,+I\cdot\mathbf{b}) \;+\; \frac{1}{2}(\,+I\cdot\mathbf{b})(\,+I\cdot\mathbf{a}) \;=\; -\,\frac{1}{2}\,\{\mathbf{ab} \;+\; \mathbf{ba}\} \;=\; -\,\mathbf{a}\cdot\mathbf{b} \;+\; 0\,, \tag{30}$$

because the orientation λ^k of S^3 is a fair coin. Here the last equality follows from the definition of the inner product.

References

[1] C. Doran and A. Lasenby, *Geometric Algebra for Physicists* (Cambridge University Press, Cambridge, 2003), page 33.
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[3] J. Christian, *Local Causality in a Friedmann-Robertson-Walker Spacetime*, `http://arXiv.org/abs/1405.2355` (2014).