Article Title: Eight-dimensional octonion-like but associative normed division algebra

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To: Prof. Scott Chapman, Editor-in-Chief, *Communications in Algebra*

Re: Defense against reports of concern about the article 10.1080/00927872.2020.1791899

Date: October 2, 2020

Dear Professor Chapman,

Thank you for allowing me to respond to the post-publication concerns raised about the above article.

Please note that to date no one—including both of your reviewers—has found a mistake in my calculation of the norm relation (46), as presented in the article. That is because there is no mistake. The calculation is quite elementary and can be verified easily. I am well aware of both Hurwitz's theorem and Frobenius' theorem. But the results and claims presented in my article do not contradict either of these theorems, as I have stated and explained in the article.

On pages 2 to 4 of this document I provide point-by-point response to the comments by the pre- and post-publication reviewers. On pages 5 and 6 of the document I also include a simplified outline of the proof of the norm relation (46).

I hope you will recognize from my response that, since no mistake in the article—either in analysis or in method—has been identified by the reviewers that invalidate the results presented or conclusions reached in it, there is no basis for retraction of the article, according to the criteria set out by Taylor & Francis publication policy I have quoted above.

To reiterate my main point, the results presented in my article are correct. I stand by them as they are presented.

Finally, if the paper is retracted, then I request that my response is also published along with the retraction notice.

Sincerely,

Joy Christian

Reviewer # 1 — pre-publication review:

The paper describes an eight-dimensional sub-algebra of the 16-dimensional associative Clifford algebra $Cl_{4,0}$ whose eight-dimensional elements respect the norm relation. One of its important features is that the corresponding 7-sphere has a topology that differs from that of octonionic 7-sphere. The paper is well-written and the results are correct and clearly presented. My only advice to the author before recommending the paper for publication is to include some information on possible applications of the main result.

Author's response: The most significant part of this original reviewer report is: "The paper is well-written and **the results are correct and clearly presented**." I followed the advice of the reviewer in the revised version and added a paragraph on existing as well as possible applications of the main result. The paper was subsequently accepted.

Reviewer # 1 — post-publication review:

Unfortunately I did not have the time to check thoroughly all computations, but I looked into the paper again and I now have concerns. I am not convinced that the norm relation (10) indeed holds for all elements of 8-dimensional algebra constructed in the paper. At least the proof presented by the author is not very convincing in this respect. I apologize for not noticing this earlier.

Author's response: This extraordinary U-turn by the reviewer sounds more like a confession under pressure than a free admission of guilt. During the two rounds of review process—lasting for six months—there was no hesitation from the reviewer in recommending the paper for publication. On the other hand, the retrospective retraction of the original assessment of the results does not seem credible. The reviewer's original firm and free assessment, "the results are correct and clearly presented", seems far more credible than "I am not convinced" and "the proof presented by the author is not very convincing." In any case, the retrospective retraction by the reviewer cannot possibly be taken as incontrovertible evidence of a mistake in the paper. The reviewer was convinced before, and very rightly so.

The bottom line: Reviewer # 1 has *not* identified a mistake in the demonstration of the norm relation (10) or (46).

Reviewer: # 2 — post-publication review:

Report on "Eight-dimensional octonion-like but associative normed division algebra"
by Joy Christian

Author's response: As we shall see, despite a more meaningful report in comparison, Reviewer # 2 also has *not* identified a mistake in the demonstration of the norm relation (10) or (46) — *i.e.*, in the main proof of the paper.

Reviewer # 2: The main error in this paper is visible not only in the title but in the abstract, which claims "We present an eight-dimensional even sub-algebra of the $2^4 = 16$-dimensional associative Clifford algebra $Cl_{4,0}$ and show that its eight-dimensional elements denoted as X and Y respect the norm relation $||XY|| = ||X||\,||Y||$, thus forming an octonion-like but associative normed division algebra." In short, the author claims to have found an 8-dimensional normed division algebra over $\mathbb{R}$.

Author's response: The reviewer has neglected to include in the above quote *the most important part* of the first sentence of the abstract. Whether or not this is deliberate, the omission leads the reviewer to make claims that are not justified. The first sentence of the abstract of the paper does not end where the reviewer has inserted a full stop. The first sentence, in fact, continues to read as follows: "..., where the norms are calculated using the fundamental geometric product instead of the usual scalar product." This is *the* important part of the first sentence. The entire argument presented in the paper stems from it. And yet, the reviewer has chosen to quote the sentence without it.

Reviewer # 2: But there is no such thing: Hurwitz's theorem says all 8-dimensional normed division algebras over the reals are nonassociative—and isomorphic to the octonions. This famous result, published in 1923, has been confirmed with a number of proofs. I've gone through several myself. Thus, the author's main result must be false.

Author's response: I am well aware of Hurwitz's theorem. It does not imply that the main result of the paper "must be false", because the norms appearing in the main result (46) are calculated using geometric product instead of inner product. Recall that inner products are employed as the essential ingredient in all proofs of Hurwitz's theorem.

Moreover, the reviewer has guessed that the "main result must be false" without providing any evidence of a mistake in either of the two proofs of the main result presented in the paper. In fact, the main result, (46), is quite correct. It is proven explicitly in the paper, providing every step of its proof in such a manner that any competent undergraduate student can reproduce it in less than an hour, especially if they are familiar with the language of Geometric Algebra.

Reviewer # 2: The author doesn't mention Hurwitz's theorem, but in an appendix he does mention Frobenius' related result that "a finite-dimensional associative division algebra over the reals is necessarily isomorphic to either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$" (his words, and correct).

Author's response: Hurwitz's original theorem of 1898 is cited in the paper as reference [7]. Hurwitz's 1923 theorem is not mentioned in the paper because, once again, the norms appearing in the main result (46) are calculated using geometric product instead of inner product. But the reviewer has chosen to ignore this fact. As is well known, inner products are employed as the essential ingredient in all proofs of both Hurwitz's theorem and Frobenius' theorem.

Reviewer # 2: This too contradicts his main result! How does he respond? He writes "Frobenius theorem suggests that those Clifford algebras that are not isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ will contain non-zero zero divisors or idempotent elements. It is therefore important to understand how the definition (b) of the norm leading to the quadratic form $\mathcal{Q}(X)$ prevents non-zero zero divisors from occurring in the algebra $\mathcal{K}^{\lambda}$."

Author's response: The purpose of appendix is to illustrate, using an explicit example, that the Frobenius' theorem does not contradict the main result presented in the paper, because—once again—the norms appearing in the main result (46) are calculated using geometric product, not inner product. This fact is explicitly captured within the definition (b) discussed in the paper. In the appendix it is explicitly proven that "the *ad hoc* coefficients chosen in (A.6) to define $\mathbf{X}$ and $\mathbf{Y}$ confined to a two-dimensional subspace of $\mathcal{K}^{\lambda}$ are not compatible with the normalization condition (A.2), and thus they are not compatible with the definition (b) of the norm used to construct the 7-sphere defined in (47). Consequently, these multivectors are not members of the set (47) that constitutes that 7-sphere."

Reviewer # 2: In fact Frobenius' theorem *proves* that all finite-dimensional associative algebras that are not isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ *must* contain zero divisors other than zero. And the author admits that his algebra contains nontrivial idempotents: in equation (A.4) he mentions two, which he calls Z_+ and Z_-. These quantities are not equal to 0 or 1 (see (A.3), and any idempotent other than 0 or 1 *must* be a nonzero zero divisor, since $x^2 = x$ implies $x(x-1) = 0$.

Yet [he] does not draw this conclusion: instead, he tries to dodge it with some unconvincing arguments.

Author's response: Far from "dodging" the issue mentioned by the reviewer, I have confronted it head-on in the appendix. The title of the appendix is: "Illustration of how the definition (b) of the norm precludes zero divisors." Thus the purpose of the appendix is to demonstrate how the definition (b) of the norm *precludes* nontrivial idempotents, such as those assumed in equations (A.3) and (A.4), from occurring in the algebra $\mathcal{K}^\lambda$. The explicitly demonstrated reason for this is that "the *ad hoc* coefficients chosen in (A.6) to define $\mathbf{X}$ and $\mathbf{Y}$ confined to a two-dimensional subspace of $\mathcal{K}^\lambda$ are not compatible with the normalization condition (A.2), and thus they are not compatible with the definition (b)."

Reviewer # 2: As a sidenote, in the abstract the author also claims "the corresponding 7-sphere has a topology that differs from that of octonionic 7-sphere". There is no such thing as a 7-sphere with a different topology than the usual 7-sphere: the unit spheres in 8-dimensional normed real vector spaces are all homeomorphic to each other.

Author's response: Since Milnor-1956 it is well known that the 7-sphere admits exotic differential structures that fail to be diffeomorphic to the standard 7-sphere. Therefore, the reviewer's claim should be taken with a pinch of salt.

But what is meant by the sentence quoted by the reviewer from the abstract is something different. In terms of the coefficients of the quaternions $\mathbf{q}_r$ and $\mathbf{q}_d$ the normalization condition $\mathbf{q}_r\mathbf{q}_d^\dagger + \mathbf{q}_d\mathbf{q}_r^\dagger = 0$ is equivalent to the constraint

$$f_{\mathcal{K}} = q_0q_7 + \lambda\, q_1q_6 + \lambda\, q_2q_5 + \lambda\, q_3q_4 = 0, \tag{1}$$

where q_0, q_1, q_2, q_3, q_4, q_5, q_6, and q_7 are coefficients of the eight-dimensional elements of $\mathcal{K}^\lambda$. This constraint reduces $\mathcal{K}^\lambda$ to the sphere S^7, thereby reducing the 8 dimensions of $\mathcal{K}^\lambda$ to the 7 dimensions of S^7 defined in (47), with the same radius $\rho = \sqrt{\varrho_r^2 + \varrho_d^2}$ as that of the standard 7-sphere, which of course satisfies a different constraint, namely

$$f_{\mathbb{O}} = q_0^2 + q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2 + q_6^2 + q_7^2 - \rho^2 = 0, \tag{2}$$

reducing the set $\mathbb{O}$ of unit octonions to the sphere S^7 made up of eight-dimensional vectors of the same fixed length ρ.

Both constraints, (1) and (2), involve the same eight variables of the embedding space $\mathrm{I\!R}^8$, namely, q_0, q_1, q_2, q_3, q_4, q_5, q_6, and q_7, giving the same seven dimensions for the sphere S^7 of radius ρ, albeit respecting different topologies.

Author's conclusion:

The bottom line is that, neither Reviewer # 1 nor Reviewer # 2 has identified a mistake in the demonstration of the norm relation (46). By ignoring the crucial fact that the entire thesis of the paper depends on using geometric product instead of inner product, all Reviewer # 2 has managed to do is *guess* that "the author's main result must be false."

The Outline of the Proof of the Norm Relation (10), or Equivalently (46):

The reason neither reviewer has been able to identify a mistake in my demonstration of the norm relation (10), or equivalently of (46), is because *there is no mistake in it.* But since no one seems to want to go through the details of the proof of the norm relation, let me present a simplified outline of it here to demonstrate how extraordinarily simple my argument is once you follow its logic to the end. The eight-dimensional elements of $\mathcal{K}^{\lambda}$ are of the form

$$\mathbf{X} = \mathbf{q}_{r1} + \mathbf{q}_{d1}\,\varepsilon\,, \tag{3}$$

where $\mathbf{q}_{r1}$ and $\mathbf{q}_{d1}$ are quaternions and ε, with $\varepsilon^2 = +1$, is a pseudoscalar that commutes with every element of $\mathcal{K}^{\lambda}$. Consequently, using the language of Geometric Algebra, the geometric product of $\mathbf{X}$ with $\mathbf{X}^{\dagger}$ works out to be

$$\mathbf{X}\,\mathbf{X}^{\dagger} \;=\; \Big(\mathbf{q}_{r1}\,\mathbf{q}_{r1}^{\dagger} \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{d1}^{\dagger}\Big) \,+\, \Big(\mathbf{q}_{r1}\,\mathbf{q}_{d1}^{\dagger} \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{r1}^{\dagger}\Big)\,\varepsilon\,. \tag{4}$$

Now it is easy to see that $\mathbf{q}_{r1}\,\mathbf{q}_{r1}^{\dagger} = \varrho_{r1}^2$ and $\mathbf{q}_{d1}\,\mathbf{q}_{d1}^{\dagger} = \varrho_{d1}^2$ are scalar radii. Therefore, the above product reduces to

$$\mathbf{X}\,\mathbf{X}^{\dagger} \;=\; \varrho_{r1}^2 + \varrho_{d1}^2 \,+\, \Big(\mathbf{q}_{r1}\,\mathbf{q}_{d1}^{\dagger} \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{r1}^{\dagger}\Big)\,\varepsilon\,, \tag{5}$$

which is evidently a sum of a scalar quantity, $\varrho_{r1}^2 + \varrho_{d1}^2$, and a non-scalar quantity, $\Big(\mathbf{q}_{r1}\,\mathbf{q}_{d1}^{\dagger} \,+\, \mathbf{q}_{d1}\,\mathbf{q}_{r1}^{\dagger}\Big)\,\varepsilon$. This is, of course, a general feature of any geometric product. It is always a sum of a scalar part and a non-scalar part. Now, traditionally, in Geometric Algebra the norm of a multivector $\mathbf{X}$ is taken to be

$$||\mathbf{X}|| = \sqrt{\mathbf{X}\cdot\mathbf{X}^{\dagger}} = \sqrt{\varrho_{r1}^2 + \varrho_{d1}^2} =: \varrho_{\mathbf{X}}\,. \tag{6}$$

The left-hand side of this equation is thus a scalar number; namely, $||\mathbf{X}||$. But there are two equivalent ways of working out this scalar number, if all one is interested in is working out the number $||\mathbf{X}||$:

(a) $||\mathbf{X}||$ = square root of the scalar part $\mathbf{X}\cdot\mathbf{X}^{\dagger}$ of the geometric product $\mathbf{X}\mathbf{X}^{\dagger}$ between $\mathbf{X}$ and $\mathbf{X}^{\dagger}$,

or

(b) $||\mathbf{X}||$ = square root of the geometric product $\mathbf{X}\mathbf{X}^{\dagger}$ with the non-scalar part of $\mathbf{X}\mathbf{X}^{\dagger}$ set to zero.

Evidently, the above two definitions of the norm $||\mathbf{X}||$ are *entirely equivalent.* They give one and the same scalar value for the norm $||\mathbf{X}||$. Therefore, given another eight-dimensional element

$$\mathbf{Y} = \mathbf{q}_{r2} + \mathbf{q}_{d2}\,\varepsilon \tag{7}$$

of $\mathcal{K}^{\lambda}$ analogous to (3), with geometric product

$$\mathbf{Y}\,\mathbf{Y}^{\dagger} \;=\; \varrho_{r2}^2 + \varrho_{d2}^2 \,+\, \Big(\mathbf{q}_{r2}\,\mathbf{q}_{d2}^{\dagger} \,+\, \mathbf{q}_{d2}\,\mathbf{q}_{r2}^{\dagger}\Big)\,\varepsilon \tag{8}$$

analogous to (5), the questions I have investigated in the paper are: (Q1) does the norm relation

$$||\mathbf{X}\mathbf{Y}|| \;=\; ||\mathbf{X}||\;||\mathbf{Y}|| \tag{9}$$

hold if we use the second definition of the norm — the definition (b) specified above, and if so, then (Q2) what are the consequences? The answer to the first question is: *Yes*, and the proof of the relation (9) is exceedingly simple.

Suppose the multivectors $\mathbf{X}$ and $\mathbf{Y}$ belonging to $\mathcal{K}^\lambda$ are normalized using the definition (b) above as follows:

$$||\mathbf{X}|| = \sqrt{\mathbf{X}\mathbf{X}^\dagger} = \varrho_{\mathbf{X}} \quad \text{with} \quad \mathbf{q}_{r1}\,\mathbf{q}_{d1}^\dagger + \mathbf{q}_{d1}\,\mathbf{q}_{r1}^\dagger = 0 \tag{10}$$

$$\text{and} \quad ||\mathbf{Y}|| = \sqrt{\mathbf{Y}\mathbf{Y}^\dagger} = \varrho_{\mathbf{Y}} \quad \text{with} \quad \mathbf{q}_{r2}\,\mathbf{q}_{d2}^\dagger + \mathbf{q}_{d2}\,\mathbf{q}_{r2}^\dagger = 0\,, \tag{11}$$

where, as we saw above, $\varrho_{\mathbf{X}} := \sqrt{\varrho_{r1}^2 + \varrho_{d1}^2}$ and $\varrho_{\mathbf{Y}} := \sqrt{\varrho_{r2}^2 + \varrho_{d2}^2}$ are fixed scalars. But it is quite easy to verify from the multiplication table (Table 1) in the paper that $\mathcal{K}^\lambda$ remains closed under multiplication. If $\mathbf{X}$ and $\mathbf{Y}$ are elements in $\mathcal{K}^\lambda$, then so is their product $\mathbf{Z} = \mathbf{X}\mathbf{Y}$. Consequently, in analogy with (10) and (11), we may consider the quantity

$$||\mathbf{X}\mathbf{Y}|| = ||\mathbf{Z}|| = \sqrt{\mathbf{Z}\mathbf{Z}^\dagger}\,, \quad \text{but } \textit{without} \text{ assuming } \mathbf{q}_{r3}\,\mathbf{q}_{d3}^\dagger + \mathbf{q}_{d3}\,\mathbf{q}_{r3}^\dagger = 0. \tag{12}$$

Now, the explicit proof worked out in Eqs. (31) to (46) in the paper unambiguously demonstrates that even in that case, using only the normalization conditions specified in the above Eqs. (10) and (11) for $\mathbf{X}$ and $\mathbf{Y}$ (respectively), the norm relation $||\mathbf{X}\mathbf{Y}|| = ||\mathbf{X}||\,||\mathbf{Y}||$ holds, because the conditions in Eqs. (10) and (11) automatically satisfy the analogous condition for $\mathbf{Z} = \mathbf{X}\mathbf{Y}$ specified in Eq. (12) that we did not assume *a priori*, allowing us to deduce that

$$\begin{aligned}
||\mathbf{X}\mathbf{Y}|| &= \sqrt{(\mathbf{X}\mathbf{Y})(\mathbf{X}\mathbf{Y})^\dagger} && (13)\\
&= \sqrt{\mathbf{X}\mathbf{Y}\mathbf{Y}^\dagger\mathbf{X}^\dagger} && (14)\\
&= \sqrt{\mathbf{X}\,\varrho_{\mathbf{Y}}^2\,\mathbf{X}^\dagger} && (15)\\
&= \left(\sqrt{\mathbf{X}\mathbf{X}^\dagger}\,\right)\varrho_{\mathbf{Y}} && (16)\\
&= \varrho_{\mathbf{X}}\,\varrho_{\mathbf{Y}} && (17)\\
&= ||\mathbf{X}||\,||\mathbf{Y}||\,, && (18)
\end{aligned}$$

and conversely

$$\begin{aligned}
||\mathbf{X}||\,||\mathbf{Y}|| &= \varrho_{\mathbf{X}}\,\varrho_{\mathbf{Y}} && (19)\\
&= \left(\sqrt{\mathbf{X}\mathbf{X}^\dagger}\,\right)\varrho_{\mathbf{Y}} && (20)\\
&= \sqrt{\mathbf{X}\,\varrho_{\mathbf{Y}}^2\,\mathbf{X}^\dagger} && (21)\\
&= \sqrt{\mathbf{X}\mathbf{Y}\mathbf{Y}^\dagger\mathbf{X}^\dagger} && (22)\\
&= \sqrt{(\mathbf{X}\mathbf{Y})(\mathbf{X}\mathbf{Y})^\dagger} && (23)\\
&= ||\mathbf{X}\mathbf{Y}||\,. && (24)
\end{aligned}$$

This is hardly a complicated or obscure proof. What is more, every concept and every mathematical step in the paper leading to the above result (which has been proven in the paper in two different ways) is clearly explained and defined. It is therefore quite wrong to ignore these proofs and claim that "...the author's main result must be false." *It is not.*